Towards a Quantum Fluid Mechanical Theory of Turbulence

D. Drosdoff, A. Widom, J. Swain
Physics Department, Northeastern University, Boston MA USA

Y.N. Srivastava
Physics Department & INFN, University of Perugia, Perugia IT

V. Parihar
Physics Department, Boston University, Boston MA USA

S. Sivasubramanian
NSF NSEC Center for High-rate Nanomanufacturing
Northeastern University, Boston MA USA

Recent studies of turbulence in superfluid Helium indicate that turbulence in quantum fluids obeys a Kolmogorov scaling law. Such a law was previously attributed to classical solutions of the Navier-Stokes equations of motion. It is suggested that turbulence in all fluids is due to quantum fluid mechanical effects. Employing a field theoretical view of the fluid flow velocity, vorticity appears as quantum filamentary strings. This in turn leads directly to the Kolmogorov critical indices for the case of fully developed turbulence.

PACS numbers: 47.27.Gs, 67.40.Vs

I. INTRODUCTION

It is often thought that fluid mechanical turbulence is the last of the great unsolved problems of classical physics[1]. Even with extensive numerical solutions of the classical Navier-Stokes fluid mechanical equations of motion, it has not been possible to prove simple experimental rules, which have been formulated for turbulent fluid flows[2]. Our purpose is to suggest that turbulence is inherently a quantum mechanical problem which cannot be solved by classical methods.

To see what is involved in a quantum mechanical picture, let us consider the role of quantum mechanics in describing vorticity and turbulence. The classical Reynolds number describing a turbulent states is defined as

$$\mathcal{R} = \frac{\mathcal{V}\mathcal{L}}{\nu} = \frac{\mathcal{V}\mathcal{L}\rho}{\eta}\ , \tag{1}$$

wherein $\mathcal{V}$ and $\mathcal{L}$ represent, respectively, the velocity and length scales of the flow, ν and η represent, respectively, kinematic and dynamic viscosity of the fluid and ρ is the fluid mass density. Turbulent flows are normally described by high Reynolds number. The number of quantum vortex filaments characterizing a flow as described by Feynman[3] reads

$$\mathcal{R}_F = \frac{\mathcal{V}\mathcal{L}}{\kappa}, \tag{2}$$

wherein the quantized vortex circulation κ for a fluid containing molecules each of mass M and total atomic mass number A is given by

$$\kappa = \frac{2\pi\hbar}{M} \approx \frac{3.96\times 10^{-3}}{A}\ \frac{\text{cm}^2}{\text{sec}}\ . \tag{3}$$

For the formation of a single quantum vortex in an otherwise circulation free fluid flow, the Feynman number $R_F \sim 1$. The ratio

$$\frac{\mathcal{R}_F}{\mathcal{R}} = \frac{\nu}{\kappa} \tag{4}$$

denotes the number of quantized filaments within a turbulent tube normally described employing *classical* vorticity. For water H_2O[4] with $A = 18$ and at room temperature and pressure,

$$\left[\frac{\mathcal{R}_F}{\mathcal{R}}\right]_{water} = \frac{\nu_{water}}{\kappa_{water}} \approx 45. \tag{5}$$

Vortex tubes in turbulent water flows may then be safely considered to be constructed from quantized vortices.

In Sec.II the quantized theory of fluid mechanics will be reviewed[5] and formulated as a Lagrangian quantum field theory. Quantized vortex filaments will then be discussed in Sec.III. In Sec.IV the general theory of quantum fluid quantization will be written in the Clebsch representation[6]. In Sec.V the statistical correlation functions of velocity and vorticity are defined and discussed. The classical Kolmogorov model[7, 8, 9] for fully developed turbulence will then be considered wherein the critical index for velocity fluctuations will be derived. In Sec.VI, the Kolmogorov scaling exponents will be derived under the assumption that quantized vortex filaments have the geometry characteristic of a self avoiding random walk. When a path is embedded in D spatial dimensions, the fractal dimension of a random walk path is two, while the fractal dimension of a *self-avoiding* random walk path is

$$d = \frac{D+2}{3},$$
$$d = 5/3 \quad \text{for} \quad D = 3. \tag{6}$$

The fractal dimension $d = 5/3$ of a random quantized vortex line uniquely determines the Kolmogorov scaling exponents for fully developed turbulent quantum fluid dynamics. In the concluding Sec.VII, the very similar nature of turbulence in the quantum superfluid phase of $^4He_2$ and $^3He{-}B$ and in ordinary fluids usually regarded as classical will be discussed. It will be argued that turbulence in fluid flows *normally regarded as classical* are in reality inherently quantum mechanical in nature.

II. QUANTUM FLUID MECHANICS

Let M_a be the mass of the a^{th} molecule at position $\mathbf{r}_a$ in a fluid. The mass density ρ and the mass current $\mathbf{J}$ operators may be defined

$$\begin{aligned} \rho(\mathbf{r}) &= \sum_a M_a \delta(\mathbf{r}-\mathbf{r}_a), \\ \mathbf{J}(\mathbf{r}) &= \frac{1}{2}\sum_a \big(\mathbf{p}_a\delta(\mathbf{r}-\mathbf{r}_a)+\delta(\mathbf{r}-\mathbf{r}_a)\mathbf{p}_a\big), \end{aligned} \tag{7}$$

wherein the momentum $\mathbf{p}_a$ conjugate to the position $\mathbf{r}_a$ obeys

$$[\mathbf{p}_a, \mathbf{r}_b] = -i\hbar\mathbf{1}\delta_{ab}. \tag{8}$$

The mass density and the mass current thereby obey the equal time commutation relations

$$\frac{i}{\hbar}\left[\mathbf{J}(\mathbf{r}), \rho(\mathbf{r}')\right] = \rho(\mathbf{r})\mathbf{grad}\ \delta(\mathbf{r}-\mathbf{r}'). \tag{9}$$

One may formally define the fluid velocity operator $\mathbf{v}$ according to

$$\mathbf{J}(\mathbf{r}) = \frac{1}{2}\big(\rho(\mathbf{r})\mathbf{v}(\mathbf{r}) + \mathbf{v}(\mathbf{r})\rho(\mathbf{r})\big) \tag{10}$$

in which case the velocity field does not commute with the mass density field; i.e.

$$\frac{i}{\hbar}\left[\mathbf{v}(\mathbf{r}), \rho(\mathbf{r}')\right] = \mathbf{grad}\ \delta(\mathbf{r}-\mathbf{r}'). \tag{11}$$

Note the consequences of the Jacobi identities which follow from Eq.(11) in the form

$$\begin{aligned} [v_i(\mathbf{r}), [v_j(\mathbf{r}'), \rho(\mathbf{r}'')]] &= 0, \\ [\rho(\mathbf{r}''), [v_i(\mathbf{r}'), v_i(\mathbf{r})]] &= 0. \end{aligned} \tag{12}$$

The mass density commutes with the velocity component commutators. By taking the curl of both sides of Eq.(11). it follows that the vorticity components also commute with the mas density,

$$\begin{aligned} &\mathbf{\Omega}(\mathbf{r}) = \mathrm{curl}\mathbf{v}(\mathbf{r}), \\ &[\mathbf{\Omega}(\mathbf{r}), \rho(\mathbf{r}')] = 0. \end{aligned} \tag{13}$$

The commutation relations between different components of the velocity read

$$\begin{aligned} \frac{i}{\hbar}\rho(\mathbf{r})\left[v_i(\mathbf{r}), v_j(\mathbf{r}')\right] &= \frac{i}{\hbar}\left[v_i(\mathbf{r}), v_j(\mathbf{r}')\right]\rho(\mathbf{r}') \\ \frac{i}{\hbar}\rho(\mathbf{r})\left[v_i(\mathbf{r}), v_j(\mathbf{r}')\right] &= \Omega_{ij}(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'), \\ \Omega_{ij}(\mathbf{r}) &= \epsilon_{ijk}\Omega_k(\mathbf{r}). \end{aligned} \tag{14}$$

Let us now consider the quantum vorticity and fluid circulation in more detail.

III. VORTEX FILAMENTS

For a given velocity potential function $\Phi(\mathbf{r})$, consider the unitary operator

$$U = \exp\left(\frac{i}{\hbar}\int \rho(\mathbf{r})\Phi(\mathbf{r})d^3\mathbf{r}\right). \tag{15}$$

Acting on a wave function Ψ of N fluid molecules,

$$U\Psi = \exp\left(\frac{iM}{\hbar}\sum_{a=1}^{N}\Phi(\mathbf{r}_a)\right)\Psi. \tag{16}$$

The unitary operator U comutes with the mass density,

$$U^\dagger\rho(\mathbf{r})U = \rho(\mathbf{r}), \tag{17}$$

but not with the velocity,

$$U^\dagger\mathbf{v}(\mathbf{r})U = \mathbf{v}(\mathbf{r}) + \mathbf{grad}\Phi(\mathbf{r}). \tag{18}$$

Eq.(18) follows directly from Eqs.(11) and (15). Let us now consider the circulation of the velocity around a closed curve C; i.e.

$$\Gamma(C) = \oint_C \mathbf{v}\cdot d\mathbf{r}. \tag{19}$$

Eqs.(18) and (19) imply the circulation operator tranformation law

$$U^\dagger\Gamma(C)U = \Gamma(C) + \oint_C \mathbf{grad}\Phi\cdot d\mathbf{r}. \tag{20}$$

The last integral on the right hand side of Eq.(20) need not vanish if the closed curve C resides in a multiply connected region of nonzero mean fluid density.

For example, suppose a vortex filament with a finite core which is empty of fluid molecules. Let C completely surround the core. From the single valued nature of quantum mechanical wave functions, in particular the velocity potential phase in

$$U\Psi \equiv \left[\prod_{a=1}^{N} e^{iM\Phi(\mathbf{r}_a)/\hbar}\right]\Psi, \tag{21}$$

one must have the quantized circulation

$$\oint_C \mathbf{grad}\Phi \cdot d\mathbf{r} = \frac{2\pi\hbar}{M} N(C) \ ,$$

$$N(C) = 0, \pm 1, \pm 2, \pm 3, \pm 4 \ldots \quad . \tag{22}$$

The central problem of quantum fluid mechanical turbulence concerns the nature of the motions of filaments with a quantized circulation in units of $\kappa = (2\pi\hbar/M)$. Let us now turn to a formal Lagrangian representation of a field theoretical formulation of fluid mechanics.

IV. CLEBSCH QUANTIZATION

Let us at first begin with the *classical fluid mechanical action principle*[10] for adiabatic flows. Later the action will be employed to quantize the theory, The results are consistent with the Landau quantization approach discussed in the above Sec.II. From a classical fluid mechanical viewpoint, the velocity field may be written in the form

$$\mathbf{v} = \mathbf{grad}\Phi + \lambda \mathbf{grad}\mu \tag{23}$$

wherein the three component velocity fields (v_x, v_y, v_z) are replaced by three scalar fields Φ, λ and μ. The vorticity may then be computed as

$$\mathbf{\Omega} = \mathrm{curl}\mathbf{v} = \mathbf{grad}\lambda \times \mathbf{grad}\mu. \tag{24}$$

Vortex lines may be pictured as the intersection between two surfaces where one of the surfaces has spatially constant λ and the other surface has spatially constant μ. In adiabatic flows, one expects that the vortex lines move with the fluid. This expectation may be implemented by allowing *three classical conservation laws*; i.e.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) &= 0, \\ \frac{\partial (\rho\lambda)}{\partial t} + \mathrm{div}\big((\rho\lambda)\mathbf{v}\big) &= 0, \\ \frac{\partial (\rho\mu)}{\partial t} + \mathrm{div}\big((\rho\mu)\mathbf{v}\big) &= 0. \end{aligned} \tag{25}$$

In terms of the total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{grad} \ , \tag{26}$$

we have the conservation laws in the more simple form

$$\begin{aligned} \frac{d\rho}{dt} &= -\rho \mathrm{div}\mathbf{v}, \\ \frac{d\lambda}{dt} &= 0, \\ \frac{d\mu}{dt} &= 0. \end{aligned} \tag{27}$$

Thus, each of the fluid particles and their λ and μ surface coordinate assignments are conserved. The spatially constant surfaces of both λ and μ move with the fluid and so do the vortex lines which are the intersection between these surfaces.

Eqs.(27) are an expression of Kelvin's circulation theorem[11] for adiabatic flows. In particular, the circulation around a closed boundary curve $C = \partial\Sigma$ of a surface Σ may be written

$$\Gamma(C) = \oint_C \mathbf{v}\cdot d\mathbf{r} = \oint_C (d\Phi + \lambda d\mu) \tag{28}$$

wherein Eq.(23) has been invoked. If Σ lies within a simply connected fluid region of non-zero classical fluid density, then Stokes theorem implies

$$\Gamma(C) = \oint_{C=\partial\Sigma} \lambda d\mu = \int_\Sigma d\lambda \wedge d\mu. \tag{29}$$

If C, Σ, λ, and μ all move with the adiabatic flow, then the circulation $\Gamma(C)$ is uniform in time in accordance with Kelvin's theorem.

Let $u(\rho, s)$ representing the fluid energy per unit mass as a function of the mass density ρ and the entropy per unit mass s. The classical fluid action may be written as

$$A = \int L dt = \int \left\{ \int \mathcal{L} d^3\mathbf{r} \right\} dt, \tag{30}$$

with an adiabatic lagrangian density

$$\mathcal{L} = \rho\left(\frac{1}{2}|\mathbf{v}|^2 - u(\rho, s)\right) + \mathcal{L}'. \tag{31}$$

In the above Eq.(31), the first two terms on the right hand side represent the usual Galilean invariant "kinetic energy minus potential energy" form while the conservation law *constraint lagrangian density* may be written

$$\mathcal{L}' = \Phi\left(\frac{d\rho}{dt} + \rho \mathrm{div}\mathbf{v}\right) - \rho\lambda\left(\frac{d\mu}{dt}\right). \tag{32}$$

The velocity potential Φ appears as a Lagrange multiplier assuring local conservation of mass, λ appears as a Lagrange multiplier assuring local conservation of μ and local conservation of λ is the result of the classical variational action equations of motion; i.e. $\delta A = 0$. Let us consider this in more detail.

Functionally differentiating the action Eq.(30) with respect to the velocity yields Eq.(23) in the form,

$$\frac{\delta A}{\delta \mathbf{v}} = \rho\left(\mathbf{v} - \mathbf{grad}\Phi - \lambda\mathbf{grad}\mu\right) = 0. \tag{33}$$

The conservation law Eqs.(27) follow from the variational action principle equations

$$\begin{aligned} \frac{\delta A}{\delta \Phi} &= \frac{d\rho}{dt} + \rho \mathrm{div}\mathbf{v} = 0, \\ \frac{\delta A}{\delta \mu} &= \lambda\left(\frac{d\rho}{dt} + \rho\mathrm{div}\mathbf{v}\right) + \rho\frac{d\lambda}{dt} = 0, \\ \frac{\delta A}{\delta \lambda} &= -\rho\frac{d\mu}{dt} = 0. \end{aligned} \tag{34}$$

Still viewing adiabatic fluid mechanics as a classical field theory, one may compute the conjugate momenta to the fields of interest employing the general field theoretical relation $\Pi_\varphi = (\delta A/\delta\dot{\varphi})$ where $\dot{\varphi} \equiv (\partial\varphi/\partial t)$. In particular, two conjugate field-momenta may be computed via

$$\Pi_\rho = \frac{\delta A}{\delta\dot{\rho}} = \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = \Phi,$$
$$\Pi_\mu = \frac{\delta A}{\delta\dot{\mu}} = \frac{\partial\mathcal{L}}{\partial\dot{\mu}} = -\rho\lambda. \tag{35}$$

The energy of a classical fluid flow in the lagrangian formalism is

$$\mathrm{E} = \int \left\{\dot{\rho}\frac{\partial\mathcal{L}}{\partial\dot{\rho}} + \dot{\mu}\frac{\partial\mathcal{L}}{\partial\dot{\mu}} - \mathcal{L}\right\} d^3\mathbf{r}, \tag{36}$$

which after parts integration yields

$$\mathrm{E} = \int \mathcal{E} d^3\mathbf{r} \ \text{ wherein } \ \mathcal{E} = \frac{1}{2}\rho|\mathbf{v}|^2 + \rho u(\rho, s). \tag{37}$$

The quantized fluid mechanical model may now be formulated as follows:

(i) There are four fields of interest ρ, Φ, λ and μ. Either field of the pair (ρ, Φ) commutes with either field of the pair (μ, λ). From Eqs.(35), we have the two conjugate field commutation relations pairs

$$\frac{i}{\hbar}\left[\Phi(\mathbf{r}), \rho(\mathbf{r}')\right] = \delta(\mathbf{r} - \mathbf{r}'),$$
$$\frac{i}{\hbar}\left[\mu(\mathbf{r}), \lambda(\mathbf{r}')\right] = \frac{\delta(\mathbf{r} - \mathbf{r}')}{\rho(\mathbf{r})}. \tag{38}$$

The fluid velocity field operator is computed via Eq.(23), employing the operator ordering

$$\mathbf{v}(\mathbf{r}) = \mathbf{grad}\Phi(\mathbf{r}) + \frac{1}{2}\{\lambda(\mathbf{r}), \mathbf{grad}\mu(\mathbf{r})\} \tag{39}$$

where $\{A, B\} \equiv AB + BA$ and with implicit point splitting for field multiplications at the same spatial point. From Eqs.(38) and (39), one may recover the Landau equal time commutation Eqs.(11), (13) and (14).

(ii) The Hamiltonian follows from Eq.(37) to be

$$H = \int \mathcal{H} d^3\mathbf{r} \ \text{ wherein } \ \mathcal{H} = \frac{1}{2}\mathbf{v}\cdot(\mathbf{1}\rho)\cdot\mathbf{v} + \rho u(\rho, s). \tag{40}$$

Note the operator ordering between the non-commuting field operators ρ and $\mathbf{v}$.

(iii) The fluid temperature T and pressure P may be computed from the thermodynamic law

$$du = Tds + \frac{P}{\rho^2}d\rho. \tag{41}$$

In terms of the enthalpy per unit mass,

$$w = \left[\frac{\partial(\rho u)}{\partial\rho}\right]_s = u + \frac{P}{\rho}, \tag{42}$$

we have the thermodynamic law

$$dw = Tds + \frac{1}{\rho}dP. \tag{43}$$

For adiabatic flows $ds = 0$; i.e.

$$\rho\ \mathbf{grad}w = \mathbf{grad}P \qquad \text{(adiabatic)}. \tag{44}$$

(iv) The equation of motion for the mass density may be derived employing

$$\dot{\rho}(\mathbf{r}) = \frac{i}{\hbar}\left[H, \rho(\mathbf{r})\right] = -\mathrm{div}\mathbf{J}(\mathbf{r}) \tag{45}$$

wherein the current density operator $\mathbf{J}$ is defined in Eq.(10). The equation of motion for the velocity field,

$$\dot{\mathbf{v}}(\mathbf{r}) = \frac{i}{\hbar}\left[H, \mathbf{v}(\mathbf{r})\right], \tag{46}$$

has the quantum operator form

$$\dot{\mathbf{v}}(\mathbf{r}) + \mathbf{grad}\big(\frac{1}{2}\mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}) + w(\rho(\mathbf{r}), s)\big)$$
$$+\frac{1}{2}\big(\mathbf{\Omega}(\mathbf{r})\times\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r})\times\mathbf{\Omega}(\mathbf{r})\big) = 0, \tag{47}$$

with the vorticity $\mathbf{\Omega} = \mathrm{curl}\mathbf{v}$. Eq.(47) is simply the quantum field theoretical version of Newton's law,

$$\rho\frac{d\mathbf{v}}{dt} = -\mathbf{grad}P \qquad \text{(classical)}, \tag{48}$$

in thinly disguised form. In Eq.(48) we have invoked Eqs.(26) and (44). Finally, the equation of motion for vorticity follows by taking the curl of Eq.(47); i.e.

$$\dot{\mathbf{\Omega}}(\mathbf{r}) = \frac{i}{\hbar}\left[H, \mathbf{\Omega}(\mathbf{r})\right],$$
$$\dot{\mathbf{\Omega}}(\mathbf{r}) + \frac{1}{2}\mathrm{curl}\big(\mathbf{\Omega}(\mathbf{r})\times\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r})\times\mathbf{\Omega}(\mathbf{r})\big) = 0. \tag{49}$$

This completes the local lagrangian formulation of quantum fluid mechanics for adiabatic flows. The quantum field theory has been well defined modulo the usual list of suspect mathematical rigor problems endemic to strongly non-linear quantum field theories describing an infinite number of degrees of freedom.

V. KOLMOGOROV CORRELATIONS

Consider the velocity fields of fully developed turbulence in a frame of reference in which the mean drift velocity are zero. The statistical correlations of the velocity field for *isotropic incompressible turbulence* ($div\mathbf{v} = 0$) may be described by

$$\mathsf{G}(\mathbf{r} - \mathbf{r}') = \frac{1}{2}\big\langle\{\mathbf{v}(\mathbf{r}), \mathbf{v}(\mathbf{r}')\}\big\rangle$$
$$\mathsf{G}(\mathbf{r} - \mathbf{r}') = \int \mathsf{S}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left[\frac{d^3\mathbf{k}}{(2\pi)^3}\right],$$
$$\mathsf{S}(\mathbf{k}) = \left(1 - \frac{\mathbf{k}\mathbf{k}}{k^2}\right)S(k), \tag{50}$$

wherein $\{a,b\} \equiv (ab+ba)$ and

$$G(|\mathbf{r}-\mathbf{r}'|) = tr\mathsf{G}(\mathbf{r}-\mathbf{r}'),$$
$$G(|\mathbf{r}-\mathbf{r}'|) = \frac{1}{2}\big\langle \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}') + \mathbf{v}(\mathbf{r}')\cdot\mathbf{v}(\mathbf{r})\big\rangle,$$
$$G(r) = \frac{1}{\pi^2}\int_0^\infty k^2 S(k)\left[\frac{\sin(kr)}{kr}\right]dk. \qquad (51)$$

The Komlmogorov energy spectral distribution $\mathcal{E}(k)dk$ is defined as the portion of the kinetic energy per unit mass in the wavenumber width dk; i.e. Eqs.(50) and (51) imply

$$\mathcal{E}(k) = \left[\frac{k^2 S(k)}{2\pi^2}\right],$$
$$G(r) = 2\int_0^\infty \mathcal{E}(k)\left[\frac{\sin(kr)}{kr}\right]dk. \qquad (52)$$

The mean squared velocity of the isotropic turbulence is thereby

$$\frac{1}{2}\left\langle |\mathbf{v}|^2\right\rangle = \int_0^\infty \mathcal{E}(k)dk. \qquad (53)$$

Vorticity correlations may be discussed in a similar manner

$$\mathsf{F}(\mathbf{r}-\mathbf{r}') = \frac{1}{2}\big\langle\{\mathbf{\Omega}(\mathbf{r}),\mathbf{\Omega}(\mathbf{r}')\}\big\rangle$$
$$\mathsf{F}(\mathbf{r}-\mathbf{r}') = \int \mathsf{S}_{\mathbf{\Omega}}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left[\frac{d^3\mathbf{k}}{(2\pi)^3}\right],$$
$$\mathsf{S}_{\mathbf{\Omega}}(\mathbf{k}) = \left(1-\frac{\mathbf{kk}}{k^2}\right)S_{\mathbf{\Omega}}(k),$$
$$S_{\mathbf{\Omega}}(k) = k^2 S(k). \qquad (54)$$

If we allow

$$F(|\mathbf{r}-\mathbf{r}'|) = tr\mathsf{F}(\mathbf{r}-\mathbf{r}'),$$
$$F(|\mathbf{r}-\mathbf{r}'|) = \frac{1}{2}\big\langle \mathbf{\Omega}(\mathbf{r})\cdot\mathbf{\Omega}(\mathbf{r}') + \mathbf{\Omega}(\mathbf{r}')\cdot\mathbf{\Omega}(\mathbf{r})\big\rangle,$$
$$F(r) = \frac{1}{\pi^2}\int_0^\infty k^2 S_{\mathbf{\Omega}}(k)\left[\frac{\sin(kr)}{kr}\right]dk, \qquad (55)$$

then the spectral theorems hold true;

$$rF(r) = 2\int_0^\infty k\mathcal{E}(k)\sin(kr)dk,$$
$$k\mathcal{E}(k) = \frac{1}{\pi}\int_0^\infty rF)r)\sin(kr)dr. \qquad (56)$$

The mean squared turbulent vorticity may also be computed as

$$\omega^2 \equiv \left\langle |\mathbf{\Omega}|^2\right\rangle = 2\int_0^\infty k^2\mathcal{E}(k)dk. \qquad (57)$$

The *classical* viscous heating rate per unit mass from the turbulent mass eddy currents may be taken to be

$$\epsilon = \frac{\eta\omega^2}{\rho} = \nu\omega^2, \qquad (58)$$

wherein η is the fluid viscosity. Kolmogorov[9] assumed that the energy spectral function depends only on the wave number k and the dissipation per unit mass ϵ, i.e.

$$\mathcal{E} = \mathcal{F}(k,\epsilon), \qquad (59)$$

and then considered the physical dimensions of the quantities involved. These dimensions (in cgs units) are

$$\dim[\mathcal{E}] = \frac{\mathrm{cm}^3}{\sec^2}\ ,$$
$$\dim[\epsilon] = \frac{\mathrm{cm}^2}{\sec^3}\ ,$$
$$\dim[k] = \frac{1}{\mathrm{cm}}\ . \qquad (60)$$

The only functional form in Eq.(59) with the correct physical dimensions of Eq.(60) is given by

$$\mathcal{E}(k) = C\left[\frac{\epsilon^{2/3}}{k^{5/3}}\right], \qquad (61)$$

where C is a dimensionless constant presumably of order unity[12]. The correlation between turbulent velocities at two different points may be defined as

$$g(\mathbf{r}-\mathbf{r}') = \big\langle|\mathbf{v}(\mathbf{r})-\mathbf{v}(\mathbf{r}')|^2\big\rangle\,,$$
$$g(r) = 2(G(0)-G(r)),$$
$$g(r) = 4\int_0^\infty \mathcal{E}(k)\left[1-\frac{\sin(kr)}{kr}\right]dk, \qquad (62)$$

wherein Eq.(52) has been invoked. Employing Eqs.(61) and (62), one finds

$$C' = 4C\int_0^\infty \frac{1}{\beta^{5/3}}\left[1-\frac{\sin\beta}{\beta}\right]d\beta,$$
$$C' = \frac{9}{5}\Gamma\left(\frac{1}{3}\right)C,$$
$$g(r) = C'(\epsilon r)^{2/3}, \qquad (63)$$

wherein the gamma function,

$$\Gamma(z) = \int_0^\infty s^z e^{-s}\frac{ds}{s}\ , \qquad (64)$$

has been invoked. Under the assumption that $g = \mathcal{G}(r,\epsilon)$ alone, the proportionality in Eq.(63), i.e. $g(r) \propto (\epsilon r)^{2/3}$ follows from the analysis of physical dimensions. From Eqs.(56) and (61) follows the Kolmogorov correlation scaling function for vorticity

$$\tilde{C} = \Gamma\left(\frac{1}{3}\right)C = \frac{5}{9}C',$$
$$F(r) = \tilde{C}\frac{\epsilon^{2/3}}{r^{4/3}}\ . \qquad (65)$$

Let us now see how the Kolmogorov scaling Eq.(65) may arise from quantum vortex filaments.

VI. QUANTUM VORTEX FILAMENTS

The scaling laws in Eqs.(61) and (63) appear to be valid for fully developed turbulence in superfluid Helium[13, 14, 15, 16] as well as in liquids normally regarded as classical in nature. Let us see why this may be so from the viewpoint of quantum fluid mechanical vortex filaments. From Eq.(22), it follows that the circulation around a single quantum fluid vortex obeys

$$\oint \mathbf{v}\cdot d\mathbf{r} = \kappa \equiv \frac{2\pi\hbar}{M}\ . \qquad (66)$$

The magnitude of fluid vorticity is related to the number of quantum vortex per unit area $\mathcal{A}^{-1}$ via $\Omega = (\kappa/\mathcal{A})$. If we examine just one quantum vortex filament and let $\mathbf{T}$ be the unit tangent vector to the quantized filament, then

$$\mathbf{\Omega} = \frac{\kappa}{\mathcal{A}}\mathbf{T}. \qquad (67)$$

Following this single quantum vortex filament, let s denote the arc length alongh the filament path so that

$$\mathbf{T}(s) = \frac{d\mathbf{r}(s)}{ds}. \qquad (68)$$

Along a single quantum vortex filament, let us consider the correlation function

$$H(s_1 - s_2) = \frac{1}{2}\left\langle \mathbf{T}(s_1)\cdot\mathbf{T}(s_2) + \mathbf{T}(s_2)\cdot\mathbf{T}(s_1)\right\rangle ; \qquad (69)$$

From Eqs.(68) and (69) it follows that the mean square distance between two points on the fillament separated by a filament length L is given by

$$R^2 = \left\langle |\mathbf{r}_f - \mathbf{r}_i|^2 \right\rangle = \int_L\int_L H(s_1 - s_2)ds_1 ds_2. \qquad (70)$$

If the filament path is of fractal dimension d, then the mean square radius varies with L according to

$$\frac{L}{\xi} = \left(\frac{R}{\xi}\right)^d \quad \Rightarrow \quad R^2 = \xi^2\left(\frac{L}{\xi}\right)^{2/d}, \qquad (71)$$

wherein ξ is a length scale along the filament core required to achieve an appreciable bending. Evidently such a length should be large on the scale of the diameter of the filament core; i.e. $\xi \gg \sqrt{\mathcal{A}}$. In order that Eqs.(70) and (71) hold true, one may write

$$H(L) = \frac{1}{2}\left[\frac{\xi}{L}\right]^{2(d-1)/d} \qquad (72)$$

Employing Eqs.(55), (67), (69) and (72), one finds for fractal dimension d

$$F_d(R) = \left(\frac{\kappa}{\mathcal{A}}\right)^2 H(L) = \frac{1}{2}\left(\frac{\kappa}{\mathcal{A}}\right)^2\left[\frac{\xi}{L}\right]^{2(d-1)/d}, \qquad (73)$$

which now reads

$$F_d(R) = \frac{1}{2}\left(\frac{\kappa}{\mathcal{A}}\right)^2\left(\frac{\xi}{R}\right)^{2(d-1)}. \qquad (74)$$

in virtue of Eqs.(71) and (73). Eq.(74), describing vorticity correlations by employing quantized vortex filaments of fractal dimension d, is the central result of this section.

If the fractal dimension of the quanized vortex is that of a self avoiding random walk, as in Eq.(6), then $F(r)$ is given by

$$F(r) = \frac{1}{2}\left(\frac{\kappa}{\mathcal{A}}\right)^2\left(\frac{\xi}{r}\right)^{4/3} \qquad (75)$$

which exhibits the Kolmogorov scaling as in Eq.(65).

VII. CONCLUSION

A quantized theory of fluid mechanics was reviewed and formulated as a Lagrangian quantum field theory. Quantized vortex filaments were discussed. A theory of the quantized fluid mechanics was written in the Clebsch representation. Finally, a model of quantized vortices was given wherein the Kolmogorov scaling law was derived.

Recent studies of turbulence in superfluid Helium indicate that turbulence in quantum fluids obeys a Kolmogorov scaling law. In this paper, it is suggested that turbulence in all fluids is due to quantum fluid mechanical effects. There have been found to be many differences[18, 19, 20] between turbulence in superfluid He and other fluids, yet more striking, are the similarities[21]. It has been found experimentally that superfluid $^4He_2$ and more recently liquid helium $^3He - B$ exhibit the Kolmogorov scaling law found in other fluids, an unexpected result. The fact that both "classical" and quantum fluids manifest the same scaling laws, suggests that all fluids need to be treated using a quantum mechanical formalism when turbulence takes place.

Turbulent flows occur when the Reynold's number becomes large, i.e., $\mathcal{R} >> 1$. In this regime, the kinematic viscosity is small. Although turbulent flows are considered highly dissipative[22], they last a long time, even after an external energy source ceases to exist. A vanishing kinematic viscosity implies that the ratio $\mathcal{R}_F/\mathcal{R} = \nu/\kappa$ is relatively small. The implication is that there are few quantum vortices in a classical bundle. The quantum vortices themselves obey the Kolmogorov 5/3 scaling law.

[1] L.D. Landau and E.M. Lifshitz, "Fluid Mechanics", Chapter III, Pergamon Press, Oxford (1987).

[2] W.D. McComb, "The Physics of Fluid Turbulence", Oxford University Press, Oxford (1990).

[3] R.P. Feynman, *Prog. Low Temp. Phys.* **1**, 17 (1955).

[4] A. Widom and Y. N. Srivastava, *Modern Physics Letters B* **4**, 1 (1990).

[5] L.D. Landau, *J. Phys. USSR* **5**, 77 (1941).

[6] A. Clebsch, *J. Reine Angew. Math* **56**, 1 (1859).

[7] A.N. Kolmogorov, *Dok. Akad. Nauk. SSSR* **30**, 299 (1941).

[8] A.N. Kolmogorov, *Dok. Akad. Nauk. SSSR* **31**, 538 (1941).

[9] A.N. Kolmogorov, *Dok. Akad. Nauk. SSSR* **32**, 16 (1941).

[10] W. Yourgrau and S. Mandelstam, "Variational Principles in Dynamics and Quantum Theory", *Dover Publication, Inc.*, p. 147 (1979).

[11] W. Thomson, "On the Vis-Viva of a Liquid in Motion", *Cambridge and Dublin Mathematics Journal*, (1849).

[12] K. R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).

[13] J. Maurer and P. Tabeling, *Europhys Lett* **43**, 29 (1998).

[14] S. R. Stalp, L. Skrbek, and Russell J. Donnelly, *Physical Review Letters* **82**, 4831 (1999).

[15] P. E. Roche, et al., *EPL* **77** 66002 (2007).

[16] D. I. Bradley, et al., *Physical Review Letters* **101**, 065302 (2008).

[17] A. J. Chorin, *Commun. Math. Phys.* **132**, p. 519 (1990).

[18] Makoto Tsubota, *J. Phys. Soc. Jpn* **77**, 1 (2008).

[19] A. Finne, et al., *Nature* **424**, 1022 (2003).

[20] M. S. Paoletti, et al., *Physical Review Letters*, **101**, 154501 (2008).

[21] W. F. Vinen, *Physical Review B* **61**, 1410 (2000).

[22] A. Tsinober, "An Informal Introduction to Turbulence", p. 17, Kluwer Academic Publishers (2001).